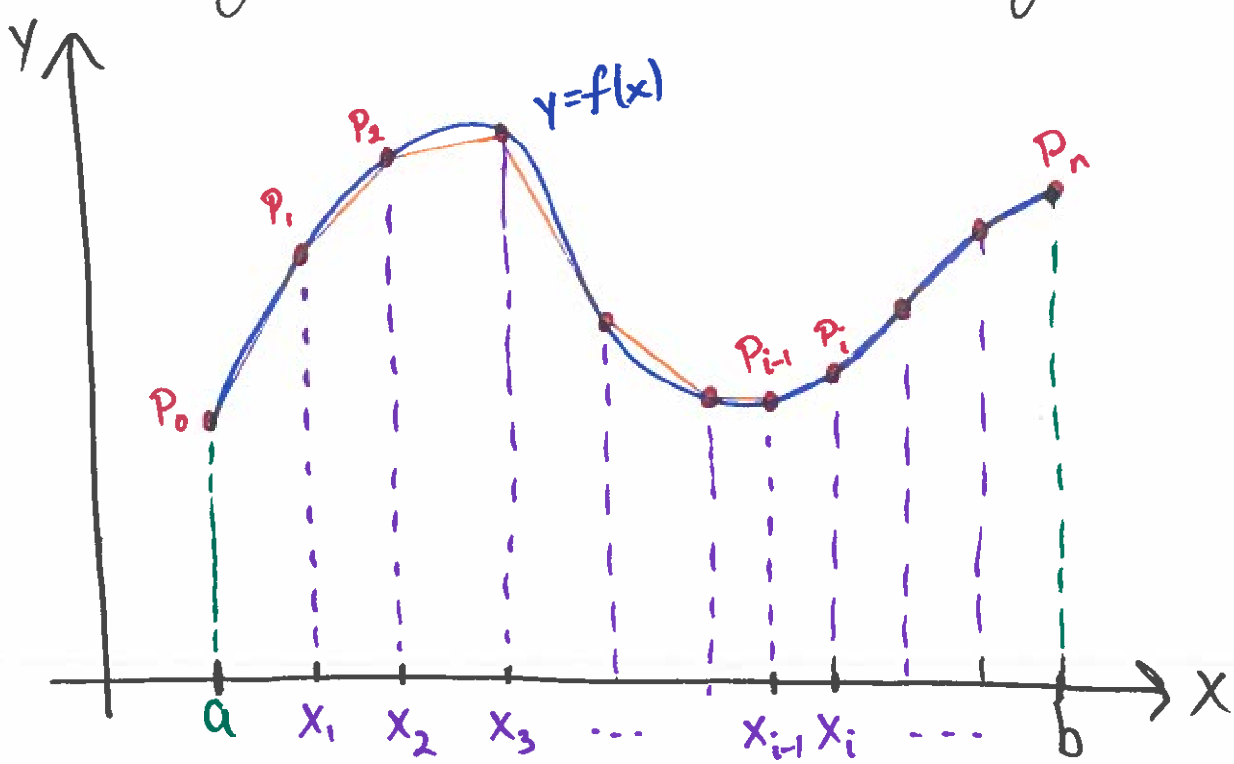
Lecture 16

8.1 - Arc Length

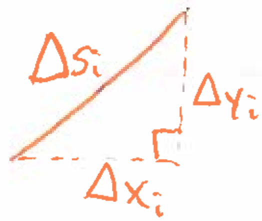
Suppose we have a curve $y=f(x)$ on an interval $[a,b]$, and that $f(x)$ is differentiable on $[a,b]$ and $f'(x)$ is continuous on $[a,b]$. We would like to find the length of this curve. We approximate the curve by straight lines and add up their lengths. In the limit, this gives the actual curve length:



Thus, we get the length as:

If we call the length of the line segment between P_{i-1} and P_i : Δs_i, then

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$



where Δx_i & Δy_i are the respective changes in x & y. Doing a clever factorization to Δs_i :

$$\Delta s_i =$$

This gives us the definition of arc length:

$$L = \int_{\text{curve}} ds = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s_i$$

(Calc III notation)

And, from this comes the definition of the arclength function

this gives the length starting at $(a, f(a))$ and ending at $(x, f(x))$

Ex: Find the length of the curve
$y = 1 + 6x^{3/2}$ for $0 \leq x \leq 1$.

Sometimes, a curve is more easily described by $x = g(y)$, $c \leq y \leq d$, where g is differentiable on $[c,d]$ and g' is continuous there as well (an easier way to say this is that g is C^1 on $[c,d]$). Then, by factoring Δy_i out of Δs_i instead of Δx_i, we get another formula for the arc length:

Ex: Find the arclength function, $s(y)$, for the curve $x = \frac{y^4}{8} + \frac{1}{4y^2}$, starting at $(\frac{3}{8}, 1)$ (and increasing in y-values).

It is possible, and, in fact, is usually the case, that these arclength integrals are difficult or impossible to compute. In these cases we use an approximation method such as Simpson's rule.

Ex: Show that the circumference of a circle of radius r is in fact $2\pi r$.